ON CONDITIONS FOR THE EXISTENCE OF PERIODIC SOLUTIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDES CONTAINING A SMALL PARAMETER

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1. Certain preliminary remarks. Consider the system of differential equations of the form

$$\frac{dx}{dt} = X(x, t) + \mu f(x, t, \mu) \qquad (1.1)$$

where x, X, f are n-dimensional vectors, and μ is a small parameter.

It is supposed that:

a) The functions X and f are defined and single-valued for all real values of t, for all values of μ lying in an interval $0 < \mu \leqslant \mu_0$, and for all x lying in an n-dimensional domain G;

b) for all x and μ in question, the functions X and f are continuous and possess the period T:

$$X(x, t + T) \equiv X(x, t), \qquad f(x, t + T, \mu) \equiv f(x, t, \mu)$$

c) the domain G is divided, by means of continuous smooth surfaces, into a finite number of subdomains G_k, in each of which, including its boundary, the function X possesses continuous second-order partial derivatives with respect to x, and the function f possesses continuous first-order partial derivatives with respect to x and μ, for $0 < \mu \leqslant \mu_0$;

d) on the surfaces separating the domains G_k (surfaces which will henceforth be referred to as discontinuity surfaces) there may occur discontinuities of the first kind of the functions X and f, or of their first-order partial derivatives with respect to x and μ as the case may be, or of the second-order partial derivatives of X with respect to x;

e) the equation of the surface of discontinuity between the domains G_k and G_{k+1} is taken to be of the form

$$\varphi_k(x) = 0 \tag{1.2}$$

It is assumed that the functions ϕ_k possess continuous second-order partial derivatives on the portions of the surfaces (1.2) which actually lie in the domain G.

It is supposed also that the generating system

$$\frac{dx_0}{dt} = X(x_0, t) \tag{1.3}$$

possesses a family of periodic solutions, depending on l independent parameters

$$x_0 = x_0(t, h_1, \ldots, h_l) \tag{1.4}$$

that all integral curves of this family pass through each of the G_k, that at all points of the intersection of one of these curves and surface (1.2) the following condition holds:

$$\frac{\partial \varphi_k}{\partial x}\frac{dx_0}{dt} \neq 0 \qquad \left(\frac{\partial \varphi_k}{\partial x} \equiv \operatorname{grad} \varphi_k\right) \tag{1.5}$$

and that the domain G contains an n-dimensional neighborhood of each point of the $(l + 1)$ dimensional manifold (1.4). From the results obtained in [1, 2] it follows that, under the hypotheses enumerated explicitly above, there exist periodic solutions of the system (1.1) in the neighborhood of certain solutions (1.4), and tending continuously to them as $\mu \to 0$. In order for this to happen, the values of the parameters $h_1, \ldots, h_l$ in the corresponding generating solutions must satisfy certain l conditions:

$$P_i(h_1, \ldots, h_l) = 0 \qquad (i = 1, \ldots, l) \tag{1.6}$$

These conditions were obtained in [1] in the form of a determinant of order $n - l + 1$, for whose actual construction there is required the knowledge of the point transformation effected by the solutions of the system (1.1). Meanwhile, in the case of equations with "smooth" right-hand sides, an integral form of conditions (1.6) has been discovered [3].

In the present paper it is shown that this (integral) form of the necessary condition for the existence of periodic solutions may be extended to the case of equations with discontinuous right-hand sides.

2. Equations for the initial conditions. The derivation of the conditions for the existence of the periodic solutions of Equation (1.1) which are near to a solution of (1.4) may be carried out without employing the method of point transformations.

Suppose that the integral curves (1.4) pass in succession through the domains G_1, G_2, ..., G_m. Since the domain G_m is closed, these curves must return again to the domain G_1. The equation of the boundary surface between G_m and G_1 is

$$\varphi_m(x) = 0$$

On each of the domains G_k the conditions for the existence and uniqueness of the solutions of the system (1.3) are fulfilled. Consequently, on each of these domains one has a general solution of this equation, depending on an initial vector C_k and on an initial instant of time t_{0k}:

$$x_{0k} = x_{0k}(t, t_{0k}, C_k) \qquad (k = 1, \ldots, m) \tag{2.1}$$

where

$$x_{0k}(t_{0k}, t_{0k}, C_k) = C_k$$

The integral curves (1.4) must, in G_k, coincide with some of the curves (2.1). On the surfaces of discontinuity the conditions of continuity and periodicity must be satisfied. If τ_k denotes the instants at which the integral curves intersect the surfaces of discontinuity, then we must have

$$x_{0k}(\tau_k, t_{0k}, C_k) - x_{0,\,k+1}(\tau_k, t_{0,\,k+1} C_{k+1}) = 0 \qquad (k = 1, \ldots, m-1) \tag{2.2}$$

$$x_{0m}(\tau_m, t_{0m}, C_m) - x_{01}(\tau_m - T, t_{01}, C_1) = 0 \tag{2.3}$$

$$\varphi_k[x_{0k}(\tau_k, t_{0k}, C_k)] = 0 \qquad (k = 1, \ldots, m) \tag{2.4}$$

Equation (1.3) has a family of solutions (1.4), therefore the system (2.2) to (2.4) must also have an l parameter family of solutions of the form

$$C_k = C_k(h_1, \ldots, h_l), \qquad \tau_k = \tau_k(h_1, \ldots, h_l) \tag{2.5}$$

Here, t_{0k} may be chosen arbitrarily in the interval $\tau_{k-1} < t_{0k} \leqslant \tau_k$. On the other hand, it is easily seen that the general solution of Equation (1.1) in G_k is given by

$$x_k = x_k(t, t_{0k}, D_k) = x_{0k}(t, t_{0k}, D_k) + \mu y_k(t, t_{0k}, D_k, \mu) \qquad (k = 1, \ldots, m) \tag{2.6}$$

where t_{0k} has the same value as in (2.2) to (2.4).

The initial vector D, at the instant τ_k' at which the integral curves return to the surfaces of discontinuity, may be determined from the equations

$$x_{0k}(\tau_k', t_{0k}, D_k) + \mu y_k(\tau_k', t_{0k}, D_k, \mu) - x_{0, k+1}(\tau'_k, t_{0, k+1}, D_{k+1}) - \\ - \mu y_{k+1}(\tau_k', t_{0, k+1}, D_{k+1}, \mu) = 0 \qquad (k = 1, \ldots, m-1) \tag{2.7}$$

$$x_{0m}(\tau_m', t_{0m}, D_m) + \mu y_m(\tau_m', t_{0m}, D_m, \mu) - x_{01}(\tau_m' - T, t_{01}, D_1) - \\ - \mu y_1(\tau_m' - T, t_{01}, D_1, \mu) = 0 \tag{2.8}$$

$$\varphi\,[x_{0k}(\tau_k', t_{0k}, D_k) + \mu y_k(\tau_k', t_{0k}, D_k, \mu)] = 0 \qquad (k = 1, \ldots, m) \tag{2.9}$$

It is not difficult to show that for sufficiently small values of μ the system (2.7) to (2.9) has solutions which are close to some of the solutions of (2.5), provided that the corresponding of the parameters h_i in (2.5) satisfy l conditions of the type (1.6). It may also be shown that if these values h_i satisfy

$$\frac{\partial (P_1, \ldots, P_l)}{\partial (h_1, \ldots, h_l)} \neq 0$$

then the solutions of the system (2.7) to (2.9) are unique and correspond to a single solution of the system (1.1). On the other hand, this proof need not be carried out in full, since it follows immediately from the results of [1,2].

3. Equations of linear approximation. Let a set of values of the parameters $h_1, \ldots, h_l$ be given. At the same time, let there be given one of the solutions of Equations (1.3), i.e. all C_k and τ_k are determined. In each of the domains G_k one may choose an initial instant in the interval $\tau_{k-1} < t_{0k} < \tau_k$, let us choose it such that

$$t_{0k} = \tau_k \tag{3.1}$$

Then, obviously, we shall have

$$x_{0k}(\tau_k, \tau_k, C_k) = C_k, \quad \left\| \frac{\partial x_{0k}(\tau_k, \tau_k, C_k)}{\partial C_k} \right\| = E_n \tag{3.2}$$

where E_n is the identity matrix of n rows and n columns.

It may be shown that the conditions (3.1) are not of use for substituting in (2.6), because it may happen that t_{0k} does not lie in the interval $[\tau_{k-1}', \tau_k']$ and that the point D_k is outside the domain G_k. On the other hand, it may readily be shown that the domain of definition of the functions X and f may be extended, preserving the conditions assuring the existence and uniqueness of solutions of the system (1.1), so as to contain a certain neighborhood of the boundary of the domain G_k, and that the point D_k will lie in this neighborhood when μ is sufficiently small.

We may now seek the solution of Equations (1.1) of the form (2.6), satisfying the conditions (2.7) to (2.9). If such a solution exists for arbitrarily chosen, sufficiently small μ, then as $\mu \to 0$ it converges continuously to the chosen generating solution, and the system (2.7) to (2.9) must then possess a solution which is close to the solution of (2.2) to (2.4), that is, the τ_k' and D_k must differ but little from the τ_k and C_k. Consequently, with an accuracy to higher-order terms, the conditions (2.7) to (2.9) must be equivalent to the following relations:

$$x_{0k}(\tau_k, \tau_k, C_k) + \left\| \frac{\partial x_{0k}(\tau_k, \tau_k, C_k)}{\partial C_k} \right\| \delta C_k + \frac{\partial x_{0k}(\tau_k, \tau_k, C_k)}{\partial t} \delta\tau_k + \tag{3.3}$$

$$+ \mu y_k(\tau_k, \tau_k, C_k, 0) - x_{0,\,k+1}(\tau_k, \tau_{k+1}, C_{k+1}) - \left\| \frac{\partial x_{0,\,k+1}(\tau_k, \tau_{k+1}, C_{k+1})}{\partial C_{k+1}} \right\| \delta C_{k+1} -$$

$$- \mu y_{k+1}(\tau_k, \tau_{k+1}, C_{k+1}, 0) - \frac{\partial x_{0,\,k+1}(\tau_k, \tau_{k+1}, C_{k+1})}{\partial t} \delta\tau_k = 0 \quad (k = 1, \ldots, m-1)$$

$$x_{0m}(\tau_m, \tau_m, C_m) + \left\| \frac{\partial x_{0m}(\tau_m, \tau_m, C_m)}{\partial C_m} \right\| \delta C_m + \frac{\partial x_{0m}(\tau_m, \tau_m, C_m)}{\partial t} \delta\tau_m + \tag{3.4}$$

$$+ \mu y_m(\tau_m, \tau_m, C_m, 0) - x_{01}(\tau_m - T, \tau_1, C_1) - \left\| \frac{\partial x_{01}(\tau_m - T_1, \tau_1, C_1)}{\partial C_1} \right\| \delta C_1 -$$

$$- \frac{\partial x_{01}(\tau_m - T, \tau_1, C_1)}{\partial t} \delta\tau_m - \mu y_1(\tau_m - T, \tau_1, C_1, 0) = 0$$

$$\varphi_k(x_{0k}) + \frac{\partial \varphi_k}{\partial x_{0k}} \left[\left\| \frac{\partial x_{0k}(\tau_k, \tau_k, C_k)}{\partial C_k} \right\| \delta C_k + \frac{\partial x_{0k}(\tau_k, \tau_k, C_k)}{\partial t} \delta\tau_k + \right. \tag{3.5}$$

$$\left. + \mu y_k(\tau_k, \tau_k, C_k, 0) \right] = 0 \qquad (k = 1, \ldots, m)$$

where $\delta\tau_k = \tau_k' - \tau_k$, $\delta C_k = D_k - C_k$. In view of (2.2) to (2.4) and (3.2), Equations (3.3), (3.4) and (3.5) may be simplified; using the notation $y_{0k}(t) = y_k(t, \tau_k, C_k, 0)$, we obtain

$$\delta C_k - \left\| \frac{\partial x_{0,\,k+1}(\tau_k, \tau_{k+1}, C_{k+1})}{\partial C_{k+1}} \right\| \delta C_{k+1} + \left[\frac{\partial x_{0k}(\tau_k, \tau_k, C_k)}{\partial t} - \right. \tag{3.6}$$

$$\left. - \frac{\partial x_{0,\,k+1}(\tau_k, \tau_{k+1}, C_{k+1})}{\partial t} \right] \delta\tau_k = \mu \left[y_{0,\,k+1}(\tau_k) - y_{0k}(\tau_k) \right] \quad (k = 1, \ldots, m-1)$$

$$\delta C_m - \left\| \frac{\partial x_{01}(\tau_m - T, \tau_1, C_1)}{\partial C_1} \right\| \delta C_1 + \left[\frac{\partial x_{0m}(\tau_m, \tau_m, C_m)}{\partial t} - \right. \tag{3.7}$$

$$\left. - \frac{\partial x_{01}(\tau_m - T, \tau_1, C_1)}{\partial t} \right] \delta\tau_m = \mu \left[y_{01}(\tau_m - T) - y_{0m}(\tau_m) \right]$$

$$\frac{\partial \varphi_k}{\partial x_{0k}}\left[\delta C_k + \frac{\partial x_{0k}(\tau_k, \tau_k, C_k)}{\partial t}\delta\tau_k\right] = -\mu\frac{\partial \varphi_k}{\partial x_{0k}} y_{0k}(\tau_k) \qquad (k = 1, \ldots, m) \tag{3.8}$$

The terms neglected here, as well as those neglected in previous formulas, are small of the second order, because the conditions imposed on the functions X insure the continuity of the derivatives of the x_{0k} with respect to all its arguments; consequently, the values of these derivatives for $\delta\tau_k = \delta C_k = 0$ and the values of these derivatives for certain mean values of the arguments (at which mean values of formulas written are exactly true) differ but little for sufficiently small μ. Further, substituting from (2.6) into (1.1) and retaining only the first-order terms in these equations, we arrive at

$$\frac{dy_{0k}}{dt} = \left\| \frac{\partial X(x_{0k}, t)}{\partial x_{0k}} \right\| y_{0k}(t) + f_{0k}(t) \qquad (k = 1, \ldots, m) \tag{3.9}$$

where

$$f_{0k}(t) = f[x_{0k}(t, \tau_k, C_k), t, 0]$$

In the terminology of [4], the differential equations (3.9), together with the conditions (3.6) to (3.8), are called the linear approximation to (1.1).

Equations (3.6) to (3.8) constitute a linear nonhomogeneous system for the determination of the unknowns δC_k and $\delta\tau_k$. The coefficients of the corresponding homogeneous system form a matrix whose determinant coincides with the Jacobian of the system (2.2) to (2.4). Therefore this homogeneous system possesses l independent solutions. In such a case, as is known, in order that there exist solutions of the nonhomogeneous system it is necessary and sufficient that the vector of order $m(n + 1)$ formed by the right-hand terms of the nonhomogeneous system be orthogonal to all l independent solutions of the adjoint homogeneous system (i.e. the homogeneous system whose matrix of coefficients is the transpose of the matrix of the coefficients of the original homogeneous system).

4. Equations of variation. Setting

$$x_k = x_{0k}(t, \tau_k, C_k) + z_k(t) \qquad (k = 1, \ldots, m) \tag{4.1}$$

and supposing that z_k is small, we obtain the equations of variation for the periodic solutions of (1.3)

$$\frac{dz_k}{dt} = \left\| \frac{\partial X(x_{0k}, t)}{\partial x_{0k}} \right\| z_k \qquad (k = 1, \ldots, m) \tag{4.2}$$

The matrix $\| \partial X/\partial x_{0k} \|$ is continuous* in the domain G_k and Equation (1.3) has the general solution (2.1), hence Equation (4.2) has, in this domain, the general solution

$$z_k = \left\| \frac{\partial x_{0k}(t, \tau_k, C_k)}{\partial C_k} \right\| A_k \qquad (k = 1, \ldots, m) \tag{4.3}$$

where A_k is an arbitrary column vector, which is small, in view of the smallness of the functions z_k.

Let us now explain how the constants A_k may be chosen in order that the variational motion (4.1) satisfy the continuity condition on the boundary of the domains G_k and the condition of periodicity with period T. Substituting from (4.3) into (4.1), and then setting (4.1) into (2.2) to (2.4), we obtain

$$x_{0k}(\tau_k^*, \tau_k, C_k) + \left\| \frac{\partial x_{0k}(\tau_k^*, \tau_k, C_k)}{\partial C_k} \right\| A_k - x_{0,\, k+1}(\tau_k^*, \tau_{k+1}, C_{k+1}) - \left\| \frac{\partial x_{0,\, k+1}(\tau_k^*, \tau_{k+1}, C_{k+1})}{\partial C_{k+1}} \right\| A_{k+1} = 0 \qquad (k = 1, \ldots, m-1) \tag{4.4}$$

$$x_{0m}(\tau_m^*, \tau_m, C_m) + \left\| \frac{\partial x_{0m}(\tau_m^*, \tau_m, C_m)}{\partial C_m} \right\| A_m - x_{01}(\tau_m^* - T, \tau_1, C_1) - \left\| \frac{\partial x_{01}(\tau_m^* - T, \tau_1, C_1)}{\partial C_1} \right\| A_1 = 0 \tag{4.5}$$

$$\varphi_k \left[x_{0k}(\tau_k^*, \tau_k, C_k) + \left\| \frac{\partial x_{0k}(\tau_k^*, \tau_k, C_k)}{\partial C_k} \right\| A_k \right] = 0 \qquad (k = 1, \ldots, m) \tag{4.6}$$

Here τ_k^* is that instant of time at which the variational integral curve (4.1) intersects the surface of discontinuity.

The system of equations (4.4) to (4.6) differs from the system (2.2) to (2.4) by terms of small order. Consequently, τ_k^* differs slightly from τ_k and, up to higher-order terms, the system (4.4) to (4.6) is equivalent to the system

$$\left\| \frac{\partial x_{0k}}{\partial C_k} \right\| A_k - \left\| \frac{\partial x_{0,\, k+1}'}{\partial C_{k+1}} \right\| A_{k+1} + \left[\frac{\partial x_{0k}(\tau_k)}{\partial t} - \frac{\partial x_{0,\, k+1}'(\tau_k)}{\partial t} \right] \Delta\tau_k = 0 \qquad (k = 1, \ldots, m-1) \tag{4.7}$$

$$\left\| \frac{\partial x_{0m}}{\partial C_m} \right\| A_m - \left\| \frac{\partial x_{01}'}{\partial C_1} \right\| A_1 + \left[\frac{\partial x_{0m}(\tau_m)}{\partial t} - \frac{\partial x_{01}'(\tau_m - T)}{\partial t} \right] \Delta\tau_m = 0 \tag{4.8}$$

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* In all matrices such as this one, each row consists of the partial derivatives of one and the same function with respect to the independent variables in question.

$$\frac{\partial \varphi_k}{\partial x_{0k}} \left[\left\| \frac{\partial x_{0k}}{\partial C_k} \right\| A_k + \frac{\partial x_{0k}(\tau_k)}{\partial t} \Delta \tau_k \right] = 0 \qquad (k = 1, \ldots, m) \tag{4.9}$$

where

$$\left\| \frac{\partial x_{0k}}{\partial C_k} \right\| \equiv \left\| \frac{\partial x_{0k}(\tau_k, \tau_k, C_k)}{\partial C_k} \right\|, \qquad \left\| \frac{\partial x'_{0,\,k+1}}{\partial C_{k+1}} \right\| \equiv \left\| \frac{\partial x_{0,\,k+1}(\tau_k, \tau_{k+1}, C_{k+1})}{\partial C_{k+1}} \right\|$$

$$\left\| \frac{\partial x_{01}'}{\partial C_1} \right\| \equiv \left\| \frac{\partial x_{01}(\tau_m - T, \tau_1, C_1)}{\partial C_1} \right\|, \qquad \frac{\partial x_{0k}(\tau_k)}{\partial t} \equiv \frac{\partial x_{0k}(\tau_k, \tau_k, C_k)}{\partial t}$$

$$\frac{\partial x_{0,\,k+1}(\tau_k)}{\partial t} = \frac{\partial x_{0,\,k+1}(\tau_k, \tau_{k+1}, C_{k+1})}{\partial t}, \qquad \Delta\tau_k = \tau_k^* - \tau_k$$

Equations (4.9) define uniquely the Δr_k:

$$\Delta \tau_k = -\frac{1}{d\varphi_k(\tau_k)/dt} \left(\left\| \frac{\partial x_{0k}}{\partial C_k} \right\| A_k \right) \frac{\partial \varphi}{\partial x_{0k}} \qquad \left(\frac{d\varphi_k(\tau_k)}{dt} \neq 0 \text{ in view of (1.5)} \right) \tag{4.10}$$

From Equation (1.3) we obtain

$$\frac{\partial x_{0.k}(\tau_k)}{\partial t} - \frac{\partial x_{0.k+1}(\tau_k)}{\partial t} = X_k(\tau_k) - X_{k+1}(\tau_k) = -\Delta_k \tag{4.11}$$

where Δ_k denotes the jump of the function X across the surface of discontinuity.

Making use of (4.10) and (4.11) in (4.7) and (4.8), keeping in mind (3.2), we obtain

$$A_k - \left\| \frac{\partial x'_{0,\,k+1}}{\partial C_{k+1}} \right\| A_{k+1} + \Delta_k \frac{A_k \partial \varphi_k / \partial x_{0k}}{d\varphi_k(\tau_k)/dt} = 0 \qquad \begin{pmatrix} k = 1, \ldots, m \\ m + 1 \equiv 1 \end{pmatrix} \tag{4.12}$$

The system of equations* which has been obtained is equivalent to the homogeneous system which corresponds to (3.6) to (3.8). Therefore it also must have l independent solutions, which determine l continuous periodic solutions of the variational equations (4.1). It should be observed that the functions z_k themselves need not, in general, be continuous. It is only when all Δ_k are equal to zero, that is, when X is continuous on every periodic solution, that the conditions (4.12) coincide with the requirement that the functions z_k be continuous.

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* It is easy to see that (4.12), together with (4.2), coincide with the linear approximation of the equations (1.3) which is obtained in [4].

Thus we may construct in this manner a set of l periodic solutions of the equations of variation (4.2), satisfying the conditions (4.12).

Let us now consider the system of linear equations which is adjoint to (4.2)

$$\frac{du_k}{dt} + \left\| \frac{\partial X(x_{0k}, t)}{\partial x_{0k}} \right\|^* u_k = 0 \qquad (k = 1, \ldots, m) \tag{4.13}$$

and the system of linear equations

$$B_{k+1} - \left\| \frac{\partial x'_{0,\,k+1}}{\partial C_{k+1}} \right\|^* B_k + \frac{\partial \varphi_{k+1}(\tau_{k+1})}{\partial x_{0k}} \frac{(\Delta_{k+1}(\tau_{k+1}) B_{k+1})}{d\varphi_{k+1}(\tau_{k+1}) / dt} = 0 \tag{4.14}$$

Here, and in the following, the asterisk denotes the transposed matrix.

It is readily verified that the matrix of coefficients of the system (4.14) is the transpose of the matrix of coefficients of (4.12). Hence the system (4.14) also possesses l independent solutions

$$B_k^{(1)}, \qquad B_k^{(2)}, \ldots, \qquad B_k^{(l)} \qquad (k = 1, \ldots, m) \tag{4.15}$$

Let us now seek a solution of (4.13) satisfying the conditions

$$u_{k+1}^{(i)}(\tau_k) = B_k^{(i)} \tag{4.16}$$

Inserting (4.16) into (4.14), we obtain

$$u_{k+2}^{(i)}(\tau_{k+1}) - \left\| \frac{\partial x'_{0,\,k+1}}{\partial C_{k+1}} \right\|^* u_{k+1}^{(i)}(\tau_k) + \frac{\partial \varphi_{k+1}(\tau_{k+1})}{\partial x_{0k}} \frac{(\Delta_{k+1}(\tau_{k+1}) u_{k+2}^{(i)}(\tau_{k+1}))}{d\varphi_{k+1}(\tau_{k+1}) / dt} = 0$$
$$(k = 0, 1, \ldots, m-1;\ \tau_0 = \tau_m - T) \tag{4.17}$$

In view of well-known properties of the solutions of adjoint systems, we obtain

$$\left\| \frac{\partial x'_{0,\,k+1}}{\partial C_{k+1}} \right\|^* u_{k+1}^{(i)}(\tau_k) = \left\| \frac{\partial x_{0,\,k+1}}{\partial C_{k+1}} \right\|^* u_{k+1}^{(i)}(\tau_{k+1}) = u_{k+1}^{(i)}(\tau_{k+1}) \tag{4.18}$$

because

$$\left\| \frac{\partial x_{0,\,k+1}}{\partial C_{k+1}} \right\|^* = E_n$$

Equations (4.17) then become

$$u_{k+1}^{(i)}(\tau_k) - u_k^{(i)}(\tau_k) + \frac{\partial \varphi_k}{\partial x_{0k}}(\tau_k) \frac{(\Delta_k(\tau_k) \cdot u_{k+1}^{(i)}(\tau_k))}{d\varphi_k(\tau_k) / dt} = 0 \qquad (k = 1, \ldots, m) \tag{4.19}$$

Obviously, if the function X is continuous (i.e. all the $\Delta_k = 0$) then (4.19) defines a periodic solution of the system of equations (4.13).

5. Conditions for the existence of a periodic solution. Let us return to the solution of Equation (3.9). It is readily seen that this equation has as a solution (for $t \leqslant \tau_k$)

$$y_{0k} = \int_{\tau_k}^{t} \left\| \frac{\partial x_{0k}(t, \tau, C_k)}{\partial C_k} \right\| f_{0k}(\tau)\, d\tau \qquad (k = 1, \ldots, m) \tag{5.1}$$

provided that it is required that

$$\left\| \frac{\partial x_{0k}(\tau, \tau, C_k)}{\partial C_k} \right\| = E_n \tag{5.2}$$

Inserting (5.1) into the conditions (3.6) to (3.8), we obtain the following system:

$$\delta C_k - \left\| \frac{\partial x_{0,\,k+1}'}{\partial C_{k+1}} \right\| \delta C_{k+1} - \Delta_k \delta\tau_k = \mu \int_{\tau_{k+1}}^{\tau_k} \left\| \frac{\partial x_{0,\,k+1}(\tau_k, \tau, C_{k+1})}{\partial C_{k+1}} \right\| f_{0,\,k+1}(\tau)\, d\tau \tag{5.3}$$

$$(k = 1, \ldots, m-1)$$

$$\delta C_m - \left\| \frac{\partial x_{01}'}{\partial C_1} \right\| \delta C_1 - \Delta_m \delta\tau_m = \mu \int_{\tau_1}^{\tau_m - T} \left\| \frac{\partial x_{01}(\tau_m - T, \tau, C_1)}{\partial C_1} \right\| f_{01}(\tau)\, d\tau \tag{5.4}$$

$$\frac{\partial \varphi_k}{\partial x_{0k}} \left[\delta C_k + \frac{\partial x_{0k}(\tau_k)}{\partial t} \delta\tau_k \right] = 0 \qquad (k = 1, \ldots, m) \tag{5.5}$$

Taking into account that (4.16) is a solution of the corresponding homogeneous system, the conditions for the existence of the system (5.3) to (5.5) may be written thus:

$$\sum_{k=1}^{m-1} \int_{\tau_{k+1}}^{\tau_k} \left\| \frac{\partial x_{0,\,k+1}(\tau_k, \tau, C_{k+1})}{\partial C_{k+1}} \right\| f_{0,\,k+1}(\tau)\, u_{k+1}^{(i)}(\tau_k)\, d\tau \; +$$

$$+ \int_{\tau_1}^{\tau_m - T} \left\| \frac{\partial x_{01}(\tau_m - T, \tau, C_1)}{\partial C_1} \right\| f_{01}(\tau)\, u_1^{(i)}(\tau_m - T)\, d\tau = 0 \qquad (i = 1, \ldots, l) \tag{5.6}$$

Making use of (4.18), and denoting by $f_0(t)$ the function which equals f_{0k} on each domain G_k, and denoting by $u^{(i)}(t)$ the function which equals $u_k^{(i)}(t)$ in G_k we obtain the conditions for the existence of a periodic solution in the final form:

$$\int_{\tau_m - T}^{\tau_m} f_0(\tau)\, u^{(i)}(\tau)\, d\tau = 0 \qquad (i = 1, \ldots, l) \tag{5.7}$$

For the determination of the parameters $h_1, \ldots, h_l$ we obtain the system

$$P_i(h_1, \ldots, h_l) \equiv \int_0^T f_0(\tau)\, u^{(i)}(\tau)\, d\tau = 0 \qquad (i = 1, \ldots, l) \tag{5.8}$$

If the system (5.8) has a unique solution $h_1^*, \ldots, h_l^*$, i.e. if the Jacobian

$$\left[\frac{\partial(P_1, \ldots, P_l)}{\partial(h_1, \ldots, h_l)}\right]_{h_j = h_j^*} \neq 0$$

then the system (1.1) will have a unique solution for all sufficiently small values of the parameter μ, a solution which is close to the solution of the generating system.

The conditions (5.8) coincide formally with the conditions obtained in [3] for the case of equations whose right-hand sides possess continuous partial derivatives of the second order.

However, the functions $u^{(i)}$ need not be continuous in the present case; they must satisfy the conditions (4.19), which in the present case are

$$u_{s,\,k+1}^{(i)}(\tau_k) - u_{s,\,k}^{(i)}(\tau_k) + \frac{\partial \varphi_k}{\partial x_{0s}}(\tau_k)\, \frac{1}{d\varphi_k(\tau_k)/dt} \sum_{j=1}^{n} \Delta_{kj}(\tau_k)\, u_{j,\,k+1}^{(i)}(\tau_k) = 0 \tag{5.9}$$

$$(i = 1, \ldots, l,\ s = 1, \ldots, n;\ \ k = 1, \ldots, m)$$

Note. Conditions (5.7) and (5.9) still hold when the surface of discontinuity is given in terms of a periodic function (with period T) of the time

$$\varphi_k(x, t) = 0 \tag{5.10}$$

In this case, condition (1.5) must be replaced by

$$\frac{d\varphi_k}{dt} = \frac{\partial \varphi_k}{\partial x}\, \frac{\partial x_0}{\partial t} + \frac{\partial \varphi_k}{\partial t} = 0 \tag{5.11}$$

6. Quasiconservative systems. By way of an example, illustrating the preceding results, consider a system which is "close" to a conservative system

$$\begin{aligned}\dot{q}_s &= \frac{\partial H}{\partial p_s} + \mu Q_s (q_1, \ldots, q_n, p_1, \ldots, p_n, t, \mu) \\ \dot{p}_s &= -\frac{\partial H}{\partial q_s} + \mu P_s (q_1, \ldots, q_n, p_1, \ldots, p_n, t, \mu)\end{aligned} \qquad (s = 1, \ldots, n) \qquad (6.1)$$

where H is the Hamiltonian of the generating conservative system, supposed not to depend explicitly upon t, and Q_s and P_s are periodic functions of t with period T.

Let us assume that the forces acting in the generating system

$$\dot{q}_{s0} = \frac{\partial H}{\partial p_{s0}}, \qquad \dot{p}_{s0} = -\frac{\partial H}{\partial q_{s0}} \qquad (6.2)$$

are conservative forces which are discontinuous with respect to the coordinate q_k, with a discontinuity of the first kind on the surfaces

$$\varphi_k (q_1, \ldots, q_n) = 0 \qquad (6.3)$$

In this case the derivatives $\partial H / \partial q_s$ also have discontinuities of the first kind on these same surfaces.

Suppose that the system (6.2) has a periodic solution with period T. Then, in view of the explicit independence of H upon t, it also has the one parameter family of solutions

$$q_{s0} = q_{s0} (t + h), \qquad p_{s0} = p_{s0} (t + h) \qquad (6.4)$$

The adjoint system to the equations of variation

$$\dot{u}_s = -\sum_j \frac{\partial^2 H}{\partial p_j \partial q_s} u_j + \sum_j \frac{\partial^2 H}{\partial q_j \partial q_s} v_j, \quad \dot{v}_s = -\sum_j \frac{\partial^2 H}{\partial p_j \partial p_s} u_j + \sum_j \frac{\partial^2 H}{\partial q_j \partial p_s} v_j \quad (6.5)$$

possesses a family of periodic solutions

$$u_s = -\dot{p}_{s0} (t + h), \qquad v_s = \dot{q}_{s0} (t + h) \qquad (6.6)$$

as may be readily verified by direct substitution.

Let us show that the solutions (6.6) do satisfy conditions (5.9). Indeed, the equations for v_s are identically fulfilled, since

$$v_{s, k+1} - v_{s,k} = 0, \qquad \frac{\partial \varphi_k}{\partial p_s} = 0$$

The equations for the u_s are

$$\sum_{j=1}^{2n} \Delta_{kj} u_{j, k+1} = \sum_{j=1}^{n} \left[\left(\frac{\partial H}{\partial q_{j0}} \right)_k - \left(\frac{\partial H}{\partial q_{j0}} \right)_{k+1} \right] \dot{q}_{j0} = \left(\frac{d\Pi}{dt} \right)_k - \left(\frac{d\Pi}{dt} \right)_{k+1}$$

where Π is the potential energy of the system; hence

$$u_{s,\,k+1} - u_{s,\,k} + \frac{\partial\varphi_k}{\partial q_{s0}} \frac{1}{d\varphi_k / dt} \sum_{j=1}^{n} \Delta_{kj} u_{j,\,k+1}$$
$$= \dot{p}_{s0,\,k} - \dot{p}_{s0,\,k+1} + \frac{\partial\varphi_k}{\partial q_{s0}} \frac{1}{d\varphi_k / dt} \left[\left(\frac{d\Pi}{dt}\right)_k - \left(\frac{d\Pi}{dt}\right)_{k+1}\right]$$
$$= \left(\frac{\partial H}{\partial q_{s0}}\right)_{k+1} - \left(\frac{\partial H}{\partial q_{s0}}\right)_k + \left(\frac{\partial \Pi}{\partial q_{s0}}\right)_k - \left(\frac{\partial \Pi}{\partial q_{s0}}\right)_{k+1} = 0$$

because in the case under consideration the jump of $\partial H/\partial q_{s0}$ equals the jump of $\partial \Pi / \partial q_{s0}$.

Finally, therefore, the condition for the existence of a periodic solution of the system which is close to a solution of (6.4), may be written

$$\int_0^T \sum_s [P_s (q_{j0}, p_{j0}, \tau, 0)\, \dot{q}_{s0} (\tau + h) - Q_s (q_{j0}, p_{j0}, \tau, 0)\, \dot{p}_{s0} (\tau + h)]\, d\tau = 0 \qquad (6.7)$$

For a second-order system

$$\ddot{x} + F(x) = \mu f(x, \dot{x}, t, \mu) \qquad (6.8)$$

the condition (6.7) assumes the known form

$$\int_0^T f(x_0, \dot{x}_0, \tau, 0)\, \dot{x}_0 (\tau + h)\, d\tau = 0 \qquad (6.9)$$

where $x_0(t + h)$ is the family of periodic solutions of the generating system

$$\ddot{x}_0 + F(x_0) = 0$$

Condition (6.9) was obtained earlier for analytic $F(x)$ and $f(x, \dot{x}, t, \mu)$ (see e.g. [5]); it remains valid when these functions possess a finite number of discontinuities of the first kind in the domain in question.

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